A probability that your stock will hit a certain value and when

Here is a question, a couple of questions in fact, that investors often ask themselves. One is: “Will my stock ever reach a certain price?”, and the other one is: “How long is it going to take before my stock hits a specific value?”.

To answer these questions, we’ll use the reasoning, as spelt out in two exceptional YouTube videos posted by Matt DosSantos DiSorbo. The videos can be found at:

Simple random walk: Hitting probabilities https://www.youtube.com/watch?v=9XK61C_KpJk, and
Simple random walk: Expected hitting time https://www.youtube.com/watch?v=X4r5oKXrgXU&t=97s

DiSorbo created these videos with a specific reference to simple random walks. What I have done in this paper is transcribe his videos into a written format (not sure he ever published the text), but with a couple of clarifications that you might find useful if you watch the videos. The second and more important contribution of this paper is that I have used general solutions he proposes for the random walk and applied them to stock data, hence the title of this paper. After all, the stock movements are nothing but a nonstationary random walk.

We’ll cut straight to the chase and show how to figure this out, and then we’ll explain the reason why this makes sense. If you are technically minded, read about the logic that leads to this solution. If you are not, just skip the two technical sections below.

A probability that your stock will hit a certain value

First of all, we’ll use just the closing stock prices but we’ll also calculate the daily returns. Then we’ll find out how many of the daily returns have gone up (P for positive move) and how many have gone down (N for negative move). This is then converted into percentages, which is effectively the probability of going up and down. A formula for calculating the returns, where y_t is the closing price, is (see column D in Fig 1):

$$R_t = \frac{y_t - y_{t-1}}{y_{t-1}} 100 \tag{1}$$

Below in Fig. 1 is the example of BMW's daily closing prices between 4 January 2010 and 27 October 2023. We had quite a long time series of 3,518 observations and in the end, we calculated that 50.6% of the time the stocks went up and 49.4% of the time they went down.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Date	BMW Close Price	Abs Change	Returns %				P or N			Return % All			
2	04/01/2010	€ 32.05								Mean	0.05	=AVERAGE(D3:D3519)		
3	05/01/2010	€ 32.31	0.260002	0.8112387	=((B3-B2)/B2)*100			P	=IF(D3>0,"P","N")	St Dev	1.82	=STDEV.S(D3:D3519)		
4	06/01/2010	€ 32.81	0.5	1.5475085	=((B4-B3)/B3)*100			P	=IF(D4>0,"P","N")	Max	14.47	=MAX(D3:D3519)		
5	07/01/2010	€ 33.10	0.289997	0.8838677	=((B5-B4)/B4)*100			P	=IF(D5>0,"P","N")	Min	-12.97	=MIN(D3:D3519)		
6	08/01/2010	€ 32.65	0.444999	-1.344408				N	=IF(D6>0,"P","N")	No. of P	1778	=COUNTIF(H3:H3519,"P")		
7	11/01/2010	€ 32.17	0.485001	-1.485227				N	=IF(D7>0,"P","N")	No. of N	1739	=COUNTIF(H3:H3519,"N")		
8	12/01/2010	€ 31.24	0.934997	-2.906425				N	=IF(D8>0,"P","N")	% of P	50.6	=K6/(K6+K7)*100		
9	13/01/2010	€ 31.42	0.189998	0.6082856				P	=IF(D9>0,"P","N")	% of N	49.4	=K7/(K6+K7)*100		
10	14/01/2010	€ 31.89	0.465	1.4797137				P						
11	15/01/2010	€ 31.63	0.26	-0.815303				N			Abs Changes All			
12	18/01/2010	€ 32.10	0.469999	1.485928				P		Mean	0.94			
13	19/01/2010	€ 32.43	0.330002	1.0280437				P						
3515	23/10/2023	€ 93.56	0.24	-0.255864				N						
3516	24/10/2023	€ 92.80	0.76	-0.812313				N						
3517	25/10/2023	€ 91.23	1.57	-1.69181				N						
3518	26/10/2023	€ 88.68	2.55	-2.795133	=((B3518-B3517)/B3517)*100			N	=IF(D3518>0,"P","N")					
3519	27/10/2023	€ 88.43	0.25	-0.281912	=((B3519-B3518)/B3518)*100			N	=IF(D3519>0,"P","N")					
3520			=ABS(B3519-B3518)											

Fig. 1

Fig 2 shows the graph of the BMW daily closing values in this period, which is some 14 years.

Fig. 2

Unfortunately, very often a longer history is not necessarily a good indication of where the current price is heading. For this reason, a shorter time series is a better option to calculate the percentage of Ps and Ns (positive and negative daily moves). To demonstrate the point, in Fig. 3 we took the same BMW stocks, but between January 2016 and December 2019. Still some 1,015 trading days, i.e. observations in this time series. In this period of time, we had only 47.7% daily closing stock prices going up (P) and 52.3% going down (N). Let's call this a probability of prices going up as $p = 0.477$ and a probability of going down as $q = 0.523$.

We'll use this example to demonstrate the method. We'll pretend that we do not know what happened before January 2016, nor what happened after December 2019. Our challenge is to establish how likely is that the BMW's price will ever again hit the value of €100.00.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Date	BMW Close Price	Abs Change	Returns %				P or N			Return % 2016-2019			
2	04/01/2016	€ 92.25								Mean	-0.01	=AVERAGE(D3:D1016)		
3	05/01/2016	€ 91.82	0.43	-0.466125	=((B3-B2)/B2)*100			N	=IF(D3>0,"P","N")	St Dev	1.41	=STDEV.S(D3:D1016)		
4	06/01/2016	€ 88.78	3.040001	-3.310827	=((B4-B3)/B3)*100			N	=IF(D4>0,"P","N")	Max	4.78	=MAX(D3:D1016)		
5	07/01/2016	€ 85.44	3.339997	-3.762105	=((B5-B4)/B4)*100			N	=IF(D5>0,"P","N")	Min	-7.53	=MIN(D3:D1016)		
6	08/01/2016	€ 83.44	2	-2.340824				N		No. of P	484	=COUNTIF(H3:H1016,"P")		
7	11/01/2016	€ 83.14	0.300003	-0.359543				N		No. of N	530	=COUNTIF(H3:H1016,"N")		
8	12/01/2016	€ 84.85	1.709999	2.0567705				P		% of P	47.7	=K6/(K6+K7)*100		
9	13/01/2016	€ 82.89	1.959999	-2.309958				N		% of N	52.3	=K7/(K6+K7)*100		
10	14/01/2016	€ 80.11	2.779998	-3.35384				N						
11	15/01/2016	€ 78.02	2.090004	-2.608918				N			Abs Changes 2016-2019			
12	18/01/2016	€ 78.10	0.080001	0.1025391				P		Mean	0.79			
13	19/01/2016	€ 78.86	0.760003	0.9731153				P						
1012	19/12/2019	€ 74.10	1.060006	-1.410333				N						
1013	20/12/2019	€ 74.15	0.050004	0.0674818				P						
1014	23/12/2019	€ 73.54	0.610001	-0.822658				N						
1015	27/12/2019	€ 73.51	0.029999	-0.040793	=((B1015-B1014)/B1014)*100			N	=IF(D1015>0,"P","N")					
1016	31/12/2019	€ 73.14	0.370003	-0.503337	=((B1016-B1015)/B1015)*100			N	=IF(D1016>0,"P","N")					
1017			=ABS(B1016-B1015)											

Fig. 3

The graph of BMW closing prices for the given period looks as below in Fig. 4. It shows a slight downward trend, which is why we have a greater percentage of prices going down than those going up.

Fig. 4

Before we provide the solution, let's engage in some logical thinking. It will help us understand the solution better.

If the number of Ps was greater, or equal, to 50% then it is a statistical certainty that sooner or later we will reach this magic price of €100. Why? Because a random walk is a nonstationary time series and if the probability of going up is ≥0.5, then inevitably it will climb to this number or any other number. It might take a long, long time if we pick a very high number, but the logic is that eventually, we will reach whatever number we pick. That's it! Nothing else to say or do if we have $p \geq 0.5$.

So, if $p \geq 0.5$, we do not have to do any probability calculations. We'll definitely hit the number at some point in the future, i.e. we are certain to hit the target price, so the probability is 100%.

Let's take a more interesting option, i.e. what if $p < 0.5$? In our BMW case above we have P=47.7% or $p = 0.477$, clearly less than 0.5, and subsequently N=52.3%, or $q = 0.523$.

If p is a probability of going up, and q is the opposite, i.e. a probability of going down, then it is obvious that it will always be that $p + q = 1$.

To return to our problem, if $p < 0.5$, then the solution to this problem is that the probability is calculated as:

$$P_k = (\frac{p}{q})^k \tag{2}$$

Where, as we said, p is the probability of going up, q is the probability of going down and k is the target value.

We can use a simple example. If $p = 0.4$ and $q = 0.6$ and the value of $k = 6$, for example, then the probability that this random process will hit 6 is:

$$P_6 = \left(\frac{0.4}{0.6}\right)^6 = (0.67)^6 = 0.087$$

If you plug into this formula multiple values of k and check how the results change, you will see that the probability behaves as an inverse exponential curve. In other words, for a very small value of k, probability will be close to 1 and the larger the value of k, the closer it will get to zero.

The above formula works for a random walk process where we assume the process started with some zero value. However, in our case, the last value of BMW's stock on 31 December 2019 was not zero, but €73.14 (see cell B1016 in Fig. 3). For handling stock exchange data, we need to modify how the value of *k* is calculated.

We stated that our target value is €100.00. The first step is to subtract €73.14 from €100.00, which gives us €26.86. However, this is still not *k*. We need to figure out how many steps we are likely to take from €73.14 to €100.00. The most intuitive way is to establish average daily changes in price. We will calculate absolute values. See column C in Fig 3.

On average the BMW price changed, on a daily basis, by an absolute value of 0.79 (cell K12 in Fig 3). We now need to divide 26.86 by 0.79, which gives us 34.00. This is our *k* value. To summarise:

The last price on 31/12/2019: **CP** = 73.14

Target price: **TP** = 100.00

Price differential: **PD** = TP – CP = 100.00 – 73.14 = 26.86

Average absolute daily price changes from 4/1/2016 and 31/12/2019: **AP** = 0.79

Value of ***k***: $k = \frac{PD}{AP} = \frac{26.86}{0.79} = 34.00$

Now we can use the formula to calculate the probability of reaching the value *k*:

$$P_k = (\frac{p}{q})^{\left(\frac{PD}{AP}\right)} \tag{3}$$

In our case:

$$P_{100.00} = (\frac{0.477}{0.523})^{\left(\frac{26.86}{0.79}\right)} = (0.94)^{34.00} = 0.044$$

The probability of ever hitting the price of €100.00 is 4.4%. Relatively low, but nevertheless we stand a chance. Let’s see what actually happened with BMW’s prices.

I have to add one disclaimer here. I only looked at daily closing prices. As we know, sometimes during the day a high price is much higher than the closing price. BMW prices hit €100.44 on 14 January 2022, and it was recorded as a High price of the day, but the closing price on the same day was €99.00. The closing price did not surpass €100 until 3 March 2023 when it hit €101.28. It took 1,158 days from 31 December 2019 before we hit our target price of €100.

The implication is that if the absolute average changes were higher, or if the target was lower, i.e. closer to today’s price, then the probability would be much higher. This translates nicely into some logical conclusions. Fast-growing stocks, and more volatile stocks, will register average daily changes that are much higher than stationary moving stocks. This means that they will have a higher probability of reaching a certain value than those steady and stationary stocks that do not exhibit significant growth or volatility.

All in all, equation (3) is extremely simple, derived from DiSorbo equation (2), and gives you a quick solution to calculate how likely it is that your stocks will hit some target value that you have on your mind.

Anyway, now we know how to calculate the probability that the price will reach a certain level. However, we do not know how soon this is likely to happen. We’ll dedicate the next section to this specific challenge, but before that, I’ll provide a technical explanation of how the probability formula has been derived, which is more or less a transcription of the video with some clarifications.

Hitting probabilities - technical section

A random walk process X_t starts arbitrarily from zero. It can go up and/or down, and the question is: What is the probability that it will hit some value *k*? We express this as P_k and the shorthand is:

$$P_k = P(X_t \text{ hits } k) \qquad (4)$$

This reads as: Probability P_k is the probability that a random process X_t will hit some value *k*.

As a random walk could go up and down, the probability of a random walk going up is *p* and *q* is the probability of going down. By inference $p + q = 1$, or $p = 1 - q$ and $q = 1 - p$. We are also assuming that *k* is a positive number, i.e. $k > 0$.

P_1 is the probability of a random walk making the first move, and in order to reach the value *k* we will have to take *k* steps, which effectively means that

$$P_k = P_1^k \qquad (5)$$

So, in order to calculate P_k, we first need to find a way to calculate P_1. How do we calculate the first step P_1?

We can say that P_1 is:

$$P_1 = pP_1|U + qP_1|D \qquad (6)$$

In other words, the probability of *p* that the series will go up (U) and a probability of *q* that it will go down (D).

Let's now assume that the random walk already moved by one step (the first one) and it went up. This means that we are already at P_1, and we went up, which means that $P_1|U$ is one. This could be expressed in a shorthand as:

$$P_1 = p + qP_1^2 \qquad (7)$$

The second part of the above equation (qP_1^2), is in fact qP_2, but it is more elegant to express it as qP_1^2.

We rearrange (7) this to:

$$qP_1^2 - P_1 + p = 0. \qquad (8)$$

We now have a quadratic equation with two possible solutions:

$$\frac{1 \pm \sqrt{1-4qp}}{2q} \qquad (9)$$

We'll try to simplify this expression (9). Remember that:

$$1 = p + q \qquad (10)$$

If we square (10), we get:

$$1^2 = (p + q)^2 \qquad (11)$$

Equation (11) can be expanded into:

$$1 = p^2 + q^2 + 2pq \qquad (12)$$

By subtracting $4pq$ from both sides, we get:

$$1 - 4pq = p^2 + q^2 - 2pq \tag{13}$$

This means that:

$$1 - 4pq = (p - q)^2 \tag{14}$$

Now we substitute $(p\text{-}q)^2$ in the solution $\frac{1 \pm \sqrt{1-4qp}}{2q}$, i.e. equation (9) by replacing 1-4qp. This gives us a different version of equation (9) that now looks:

$$\frac{1 \pm (p-q)}{2q} \tag{15}$$

This enables us to come up with a solution.

For $p \geq q$:

$$\frac{1-(p-q)}{2q} = \frac{1-p+(1-p)}{2q} = \frac{2q}{2q} = 1 \tag{16}$$

A solution in equation (16) by definition applies to $p \geq 0.5$.

For $p < q$, the solution is:

$$\frac{1+(p-q)}{2q} = \frac{1+p-(1-p)}{2q} = \frac{2p}{2q} = \frac{p}{q}. \tag{17}$$

Equation (17) by definition applies to $p < 0.5$.

As we already indicated, effectively, if $p \geq 0.5$ we are 100% certain that we will hit the value k, therefore, no point in doing any calculations. However, if $p < 0.5$, the probability is generally p/q, as per equation (17). This is essentially a solution for P_1:

$$P_1 = \frac{p}{q} \tag{18}$$

From there, as per equation (5) a solution to finding the probability of reaching the value k is therefore calculated as:

$$P_k = \left(\frac{p}{q}\right)^k \tag{19}$$

Hope this explains how equation (2) is derived. A very elegant and simple solution, but remember, it only works when $p < q$.

Calculating how long it would take for your stock to hit a certain value

Although it might be very valuable to calculate the probability of your stocks hitting some value, very often you would want to know how long it is going to take to hit this value. This is a different problem and we'll use the same source as stated at the beginning to describe the solution.

However, before we do this, let's offer some reasoning related to this problem.

If you have a probability of 0.1, i.e. $p = 0.1$, this is identical to saying that you have a 10 in 100 chance of this event happening. What if an event can only happen once a day, like closing values of shares that are declared only once a day? In this case, the event is bound to the time, specifically here to a day as the measure of time. Using the same terminology, we can say that given that $p = 0.1$, the event will happen on any 10 days out of a hundred.

In case you did not notice, we established a link between probability and time. If we take an inverse of probability, we get the number of days, out of 100, that the event is likely to happen. Below is a small table with some reciprocals.

Prob	0.001	0.01	0.1	0.5	0.75	0.9	0.999
1/Prob	100000	10000	1000	200	133.333	111.1111	100.1001

The reciprocal values are multiplied by 100 to get them standardised on the 100-day horizon. One way to read these numbers is as follows, and let's take $p = 0.75$: "If $p = 0.75$, this means that the event will happen on any 75 days out of 100. It is also with 100% certainty that this event will happen 100 times within 133.3 days in total." A bit confusing, right?

The probability we calculated for our BMW case was $p = 0.044$. If we take a reciprocal of this number and multiply it by 100, we get 2272.7, or 2273 days in round numbers. This means that we can claim with 100% certainty that BMW shares will hit €100 at least 100 times within the next 2273 days, though we do not know exactly on which ones of these 2273 days it is likely to happen. This does not make the reciprocal values very useful for scenarios such as these. However, there is a better way. We'll go to the second video we mentioned at the beginning. It offers a more promising solution.

Let T_k be the time needed for X_t to hit some value k. As before, X_t is a random walk. Remember that T_k is also a random variable, which means that in different runs it might take us a different amount of time to hit the value k. Because T_k is random, there is not one answer to this. The best option is to look for the expected time i.e. $E(T_k)$, which is another phrase for an average time. An average time implies would take, typically, if we run thousands of simulations. Before we start, remember again that k needs to be a positive value, i.e. $k > 0$. The same logic we will follow below could apply for $k < 0$, but everything needs to be reversed.

Let's start with the simple solution first, although not too helpful. If $p = q$, or if $p < q$ (i.e. $p \leq q$), then for both cases $E(T_1) = \infty$, where $E(T_1)$ is the expected time to hit the value k. In other words, if $p = q$, or if $p < 0.5$, i.e. both are 0.5 or p is less than 0.5, it might take infinite time to reach the value k. Well, it might not, but technically this is correct as we are talking about a probability.

For $p > 0.5$, the solution is:

$$E(T_k) = \frac{k}{p-q} \tag{20}$$

We cannot use our previous BMW example, because we had $p < q$. We'll use another example. This will be MCD from 1 June 2020 until 31 May 2022. In total 505 daily closing prices (see Fig. 5). The last value on 31 May 2022 was $252.21. This period produced $p = 0.52$ and $q = 0.48$. Our target value is $290. The question is: How many days before MCD shares hit this value?

	A	B	C	D	E	F	G	H	I
1	Date	MCD Close Price	Abs Change	Returns %	P or N		Return % June 2020 - May 2022		
2	01/06/2020	187.410004				Mean	0.07	=AVERAGE(D3:D506)	
3	02/06/2020	187.589996	0.179992	0.0960418	N	St Dev	1.13	=STDEV.S(D3:D506)	
4	03/06/2020	193.289993	5.699997	3.03854	P	Max	3.82	=MAX(D3:D504)	
5	04/06/2020	193.240005	0.049988	-0.025862	P	Min	-4.87	=MIN(D3:D506)	
6	05/06/2020	197.160004	3.919999	2.0285649	N	No. of P	260	=COUNTIF(E3:E506,"P")	
7	08/06/2020	202.649994	5.48999	2.7845353	P	No. of N	244	=COUNTIF(E3:E506,"N")	
8	09/06/2020	199.520004	3.12999	-1.54453	P	% of P	51.6	=G6/(G6+G7)*100	
9	10/06/2020	195.800003	3.720001	-1.864475	N	% of N	48.4	=G7/(G6+G7)*100	
10	11/06/2020	187.509995	8.290008	-4.233916	N				
11	12/06/2020	189.169998	1.660003	0.8852877	N		Abs Changes All		
12	15/06/2020	189.490005	0.320007	0.1691637	P	Mean	1.91		
857	23/10/2023	$ 254.75	3.36	-1.301771	N				
858	24/10/2023	$ 257.27	2.52	0.9892051	N				
859	25/10/2023	$ 258.03	0.76	0.2954095	P				
860	26/10/2023	$ 255.83	2.2	-0.852614	P				
861	27/10/2023	$ 255.76	0.07	-0.027362	N				

Fig. 5

The value of k was calculated as before:

The last price on 31/05/2022: **CP** = 252.21

Target price: **TP** = 290.00

Price differential: **PD** = TP – CP = 290.00 – 252.21 = 37.79

Average absolute daily price changes from 01/06/2020 and 31/05/2022: **AP** = 1.91

Value of ***k***: $k = \frac{PD}{AP} = \frac{37.79}{1.91} = 19.79$

From there, and using equation (20), the expected number of days to hit the value of $290 is 495 days:

$E(T_{290.00}) = \frac{19.79}{0.52-0.48} = 494.63$

What happened in reality? MCD closing shares hit the value of $290.91 on 18 April 2023, which is 322 days from 31 May 2022, which we used as our current date. Although we have a discrepancy between 495 and 322 days, we are more or less in the right ballpark. Let us not forget that this number of 495 is an expected value (an average value), which means that in the infinite number of theoretical runs, sometimes it might take less or more time, but this is an average.

We'll again provide a technical explanation of how these formulas have been derived.

Hitting time – technical section

Just to repeat, T_k is the time needed for X_t to hit some value k. X_t is a random walk. Remember what we said that T_k is also a random variable, which means that in different runs it will take a different amount of time to hit the value k.

In order to get to any k positive value, we need to take the first step to get to the first value. From this step we need to take the next one and the next one, etc., until we reach the value k. Different combinations of these steps are the total time that will take us to reach the value k. We can express this more compactly using some maths shorthand.

The expected time to hit k is calculated as:

$$E(T_k) = E(T_1) + E(T_1) + \ldots + E(T_1) \qquad (21)$$

i.e. $E(T_1)$ is multiplied k times. The other way to write this is:

$$E(T_k) = k\, E(T_1) \qquad (22)$$

As before, p is the probability of going up and q is the probability of going down at every step. This implies that $p + q = 1$, or $q = 1 - p$. If $p = q$ then each step has a 50:50 chance to go up or down, i.e. $p = q = 0.5$. If you experiment with different segments of the random walk, or different lengths of time series, you will see that this will vary from segment to segment. Just like a random walk continuously changes, so does the probability at every step of either going up or down in the next step.

Let's start with finding $E(T_1)$, as this is the key to finding $E(T_k)$. We'll take the first step and assume we went up. In this case:

$$E(T_1) = 1 + p(0) + q\,E(T_2) \qquad (23)$$

The number 1 implies that we have already taken the first step. The second term $p(0)$ is the probability to go up. As we already made a step up, this makes the probability of going up equal to zero (because we are already up). Essentially, because we are already up, the probability is zero and we can eliminate this from our equation.

The second term in equation (23) is q $E(T_2)$, which is the probability that we went down and now we need to go up. From this down position, we need first to climb back to the original position of T_1 and then to T_2. Because of that, instead of saying q $E(T_2)$, we can say 2q $E(T_1)$.

So, our equation now looks as:

$$E(T_1) = 1 + 2q\,E(T_1) \qquad (24)$$

Before we proceed, let's assume that $p = q = 0.5$. In this case:

$$E(T_1) = 1 + 2 \times 0.5\,E(T_1), \qquad (25)$$

This yields:

$$E(T_1) = 1 + E(T_1) \qquad (26)$$

This is formally illogical, so the actual answer is that if $p = q$, then:

$$E(T_1) = \infty \qquad (27)$$

In other words, if $p = q$, it might take infinite time to reach the value k (it might not, but this is a probability). Although most of the cases will take a finite amount of time, because some might take an infinite amount of time, the expected value is an infinite amount of time. Strange, but true.

Let's look at other cases where $p \neq q$. We'll take equation (24), which is $E(T_1) = 1 + 2q\,E(T_1)$ and subtract $2q\,E(T_1)$ from both sides. Now we have:

$$E(T_1) - 2q\,E(T_1) = 1 \qquad (28)$$

From there:

$$(1 - 2q)\,E(T_1) = 1 \qquad (29)$$

And finally, to extract $E(T_1)$, we have:

$$E(T_1)=\frac{1}{1-2q} \qquad (30)$$

If we replace $q = 1 - p$, we get:

$$E(T_1)= \frac{1}{p-q} \qquad (31)$$

How did we get from $1 - 2q$ to $p - q$ in the denominator? Well, first of all:

$$\begin{aligned} 1 - 2q &= 1 - 2(1 - p) \qquad (32)\\ &= 1 - 2 + 2p \\ &= 2p - 1, \text{ as } p + q = 1, \text{ then} \\ &= 2p - (p + q) \end{aligned}$$

$$= 2p - p - q$$

Which finally produces:

$$1 - 2q = p - q \qquad (33)$$

OK, so equation (31) $E(T_1) = \frac{1}{p-q}$ is a solution for $p \neq q$. The challenge with this formula is that if $p < q$, then we have a negative answer, which does not make sense. In this case, we are back to the solution where $p = q$, which is $E(T_1) = \infty$. In summary, in both cases if $p \leq q$, then $E(T_1) = \infty$.

For $p > 0.5$, we can use equation (31) $E(T_1) = \frac{1}{p-q}$. As $E(T_1)$ was just a stepping stone to find $E(T_k)$, the final solution is the equation (20), which is: $E(T_k) = \frac{k}{p-q}$.

Hope this explains clearly enough how the final solution is derived.

Just an extra bit

I picked up this great tip by Prof Jonathan Rougier from the website:

http://cabot-institute.blogspot.com/2016/12/converting-probabilities-between-time.html , and I thought it would be a waste not to share it. To a degree, it is related to our first case in this paper.

The scenario is as follows. We have a probability of 0.02 (2%) that our price will reach a certain level at least once in the next five years. If this was the case and if we were interested in the probability of the same price reaching the same level at least once in the next year, then this is easily calculated as 0.02/5=0.04, which is 0.4%.

Following the same logic, you could say that the probability of the same price hitting the same value at least once in 20 years is 0.02×4=0.08 or 8%. Why multiply by 4? Because 0.02 is the probability for 5 years and 20 years is a multiple of 4 for 5 years probability. You can check that this is correct if you multiply 0.004 (a yearly probability) by 20, which will also give you 0.08.

You would think that this is all OK and that you can use this principle for any level of probability. Prof Rougier's argument is that this is not always correct, which is the reason why what we are going to show below is so valuable. If either the original or newly calculated probability is larger than 0.1, then the logic we followed in the previous two paragraphs is NOT correct. The scaling of probabilities to different time intervals follows different rules.

The proper formula to scale the probability to a different time horizon is as follows:

$$P_{New} = 1 - (1 - P_{Orig})^{\frac{New\ time}{Orig\ time}} \qquad (34)$$

Let's use an example and let's say that we have a probability of 0.055 (5.5%) that a price will reach a certain level at least once in the next 5 years. What is the probability that it will happen again at least once over the next 20 years? The variables are:

$P_{orig} = 0.055$

Orig time = 5

New time = 20

To calculate this new probability P_{New} we use equation (34):

$$P_{20} = 1 - (1 - 0.055)^{\frac{20}{5}} = 1 - 0.945^4 = 1 - 0.797494 = 0.20$$

If the probability for the first 5 years was 0.055, then the probability for the next 20 years is 0.2, or 20%.

Note that if we used the basic rule that we used at the beginning of this section, the probability would be 0.22, which is not correct.

You might think that 0.2 and 0.22 represent only a marginal difference. However, imagine that you have a probability of 0.3 of a price hitting a certain value next year. Let's say you want to know the probability of hitting the same value in the next 5 years. If you multiply 0.3 by 5, you get 1.5, which is nonsense as you cannot have a probability over 1. The correct answer is 0.83:

$$P_5 = 1 - (1 - 0.3)^{\frac{5}{1}} = 0.8313$$

Hope this little trick makes sense and gets the visibility it deserves.

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Autumn 2023